

Lecture 9

02/13/2019

Review of Magnetostatics (Cont'd)

We note that the magnetic dipole moment of a localized current distribution does not depend on the choice of origin:

$$\vec{x} \rightarrow \vec{x} + \vec{R} \Rightarrow \vec{m} \rightarrow \vec{m} + \frac{1}{2} \int \vec{R} \times \vec{J}(\vec{x}') d\tau' = \vec{m} + \frac{1}{2} \vec{R} \times \int \vec{J}(\vec{x}') d\tau' = \vec{m}$$

(localized current)

This is consistent with the absence of magnetic monopoles; and hence magnetic dipole being the first non-vanishing multipole, as we saw in the case of electrostatics.

Considering  $N$  point charges  $q_1, \dots, q_N$  moving with respective velocities  $\vec{v}_1, \dots, \vec{v}_N$ , we have:

$$\vec{m} = \frac{1}{2} \sum_{i=1}^N \vec{x}_i \times q_i \vec{v}_i = \sum_{i=1}^N \frac{q_i}{2m_i} (\vec{x}_i \times m_i \vec{v}_i) = \sum_{i=1}^N \frac{q_i}{2m_i} \vec{L}_i$$

Including the spin angular momentum, we have:

$$\vec{m} = \sum_{i=1}^N \frac{q_i}{2m_i} \vec{L}_i + \sum_{i=1}^N \frac{q_i}{2m_i} g_s \vec{S}_i = \sum_{i=1}^N \frac{q_i}{2m_i} (\vec{L}_i + g_s \vec{S}_i)$$

$g_s \approx 2$

Now, let us consider a localized current distribution in an external magnetic field  $\vec{B}_{ent}(\vec{x})$ . Expanding the external field as a Taylor series about an origin chosen within the current volume, we have:

$$\vec{B}_{ent}(\vec{x}) = \vec{B}_{ent}(0) + (\vec{x} \cdot \vec{\nabla}) \vec{B}_{ent}(0) + \dots$$

The force on the current distribution is:

$$\begin{aligned} \vec{F} &= \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x}) d\tau = \int \vec{J}(\vec{x}) \times \vec{B}_{ent}(0) d\tau + \int \vec{J}(\vec{x}) \times (\vec{x} \cdot \vec{\nabla}) \vec{B}_{ent}(0) d\tau + \dots \\ &= \int \vec{J}(\vec{x}) d\tau \times \vec{B}_{ent}(0) + \int \vec{J}(\vec{x}) \times (\vec{x} \cdot \vec{\nabla}) \vec{B}_{ent}(0) d\tau + \dots \end{aligned}$$

Writing  $\vec{F} = F_i \hat{e}_i$ , we find (to the first non-vanishing order):

$$F_i = \epsilon_{ijk} \int J_j(\vec{x}) \partial_l B_{ent,k}(0) d\tau = \epsilon_{ijk} (\vec{m} \times \vec{\nabla})_j B_{ent,k}(0)$$

Thus:

$$\vec{F} = (\vec{m} \times \vec{\nabla}) \times \vec{B}_{ent}(0) + \dots = \vec{\nabla} (\vec{m} \cdot \vec{B}_{ent}(0)) - \vec{m} \times (\vec{\nabla} \times \vec{B}_{ent}(0))$$

$$\Rightarrow \vec{F} = -\vec{\nabla} U, \quad U = -\vec{m} \cdot \vec{B}_{ent}$$

$U$  is the potential energy due to interaction of a magnetic dipole with an external magnetic field. The minimum energy is attained

when the magnetic dipole aligns with the magnetic field.

The torque on the localized distribution follows:

$$\begin{aligned}\vec{N} &= \int \vec{x} \times (\vec{J}(\vec{x}) \times \vec{B}_{\text{ext}}(\vec{x})) d\tau = \int \vec{x} \times (\vec{J}(\vec{x}) \times \vec{B}_{\text{ext}}(0)) d\tau + \dots \\ &= \int (\vec{x} \cdot \vec{B}_{\text{ext}}(0)) \vec{J}(\vec{x}) d\tau - \vec{B}_{\text{ext}}(0) \int \vec{x} \cdot \vec{J}(\vec{x}) d\tau + \dots\end{aligned}$$

It can be shown that for a localized current distribution

$\int \vec{x}_i J_i d\tau = 0$ . Also, the first term on the right-hand side may be

transformed in a similar way as we did for finding  $\vec{F}$ . Hence:

$$\vec{N} = \vec{m} \times \vec{B}_{\text{ext}}(0) + \dots$$

Note that this is consistent with  $\vec{m} \parallel \vec{B}_{\text{ext}}(0)$  for the minimum

energy configuration, as  $\vec{N} = 0$  is expected in equilibrium.

Couple of remarks are in order. First, the interaction

energy between two magnetic dipoles  $\vec{m}_1$  and  $\vec{m}_2$  can be found by

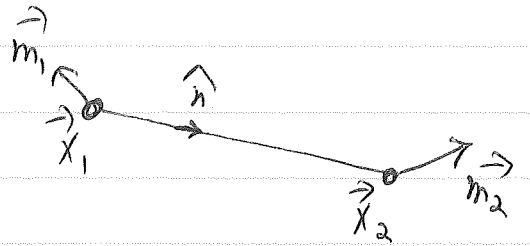
substituting the magnetic field due to one of the dipoles in

the expression \* on the previous page:

(4)

$$U = -\vec{m}_2 \cdot \vec{B}_1(\vec{x}_2) = \frac{\mu_0}{4\pi |\vec{x}_1 - \vec{x}_2|^3} [\vec{m}_1 \cdot \vec{m}_2 - 3(\vec{m}_1 \cdot \hat{n})(\vec{m}_2 \cdot \hat{n})]$$

$$= -\vec{m}_1 \cdot \vec{B}_2(\vec{x}_1)$$



Second, for a localized current distribution, we have:

$$\int_V \vec{B}(\vec{x}) d\tau = \frac{2}{3} \mu_0 \vec{m}$$

Here,  $V$  is a spherical volume that contains the entire current distribution, and  $\vec{m}$  is the magnetic dipole moment of the distribution.

### Magnetic Materials

In addition to free currents, that are carried by free charges, we also have magnetization currents carried by bound charges in the constituent atoms/molecules (or by spins, which are intrinsically quantum mechanical).

The "magnetization"  $\vec{M}(\vec{x})$  inside the material is defined as

$$\lim_{\Delta V \rightarrow 0} \frac{\Delta \vec{m}}{\Delta V}$$

which is basically the magnetic dipole density.

The vector potential due to the two currents is:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d\sigma' + \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{x}') \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} d\sigma' =$$

$$\frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d\sigma' + \frac{\mu_0}{4\pi} \int \vec{M}(\vec{x}') \times \vec{\nabla}' \left( \frac{1}{|\vec{x}-\vec{x}'|} \right) d\sigma'$$

But:

$$\vec{M}(\vec{x}') \times \vec{\nabla}' \left( \frac{1}{|\vec{x}-\vec{x}'|} \right) = -\vec{\nabla}' \times \frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

After using the identity  $\int_V \vec{\nabla} \times \vec{v} d\tau = \oint_S (\hat{n} \times \vec{v}) da$ , we find:

(for localized current)

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{x}') + \vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} d\sigma' + \frac{\mu_0}{4\pi} \oint_S \frac{\vec{M}(\vec{x}') \times \hat{n}}{|\vec{x}-\vec{x}'|} da'$$

Therefore:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}') + \vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} d\sigma'$$

Now: (as shown before)

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 (\vec{J}(\vec{x}) + \vec{\nabla} \times \vec{M}(\vec{x})) \Rightarrow \vec{\nabla} \times (\vec{B} - \mu_0 \vec{M}) = \mu_0 \vec{J} \Rightarrow \vec{\nabla} \times \vec{H} = \vec{J}, \left( \vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M} \right)$$

Henceforth, we call the  $\vec{B}$  field the "magnetic field of induction";

or simply the "magnetic induction", and the  $\vec{H}$  field the "magnetic field intensity", or simply the "magnetic field".

It is worthwhile to underline that the fundamental field is the  $\vec{B}$  field, just like the  $\vec{E}$  field in the case of electrostatics. The  $\vec{H}$  field, similar to the  $\vec{D}$  field in electrostatics, is a derived field that is introduced as a matter of convenience to take into account of the currents due to bound atomic/molecular charges.